of inertia are unstable.

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THE STABILITY OF NEUTRAL SYSTEMS IN THE CASE OF A MULTIPLE FOURTH-ORDER RESONANCE*

A.L. KUNITSYN and A.A. PEREZHOGIN

The stability of the steady motions of multiparametric systems is investigated for the critical case of N pairs of purely imaginary roots when several internal fourth-order resonances interact with each other. The earlier investigations (see the survey /1/) covered only the interaction of odd-order resonances. In general, the problem of stability when there are even-order resonances is more complicated; even in the case of the simplest, single fourth-order resonance there is no algebraic criterion of stability /2/.

1. Consider a system of $2N$-th order for the critical case of N pairs of different, purely imaginary roots $\pm\lambda_j\ (\lambda_j^2 < 0; j = 1, \ldots, N)$, which can be written in the form /1/

$$u^\cdot = \lambda u + \sum_{l=2}^{\infty} U^{(l)}(u, v), \quad v^\cdot = -\lambda v + \sum_{l=2}^{\infty} V^{(l)}(u, v) \tag{1.1}$$
$$\lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$$

where $u = (u_1, \ldots, u_N)$, $v = (v_1, \ldots, v_N)$ are complex conjugate variables and $U^{(l)}$, $V^{(l)}$ are complex conjugate vector forms of the l-th order.

Let the first $n \leqslant N$ eigenvalues of system (1.1) be connected by $\varkappa$ fourth-order resonance relations

$$\langle P_\nu, \Lambda\rangle = 0, \ \nu = 1, \ldots, \varkappa \tag{1.2}$$

Here $\Lambda = (\lambda_1, \ldots, \lambda_n)$ is the eigenvalue vector and $P_\nu = (p_{\nu1}, \ldots, p_{\nu n})$ is an integer-valued vector with relatively prime components, some of which may be equal to zero, and $|P_\nu| \equiv |p_{\nu1}| + \ldots + |p_{\nu n}| = 4$. We shall also assume that (1.2) does not give rise to other resonances of the same order and that there are not resonances of the order less than the fourth.

Following the generally accepted method of investigating the stability in the resonant, as well as in the non-resonant case, we will use a series of known variable substitutions /1/ to reduce system (1.1) to its normal form with an accuracy up to and including third-order terms. In the polar coordinates r_j, φ_j the system will have the form

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$$\frac{1}{2} r_s^{\cdot} = \sum_{\nu=1}^{\varkappa} R_\nu Q_{\nu s}(\psi_\nu) + r_s \sum_{j=1}^{N} c_{sj} r_j + \ldots \tag{1.3}$$

$$\psi_\nu^{\cdot} = \sum_{k=1}^{n} |p_{\nu k}| \Big(\sum_{l=1}^{\varkappa} \frac{R_l}{r_k} \frac{dQ_{lk}}{d\psi_l} + \sum_{j=1}^{N} d_{kj} r_j \Big) + \ldots$$

$$s = 1, \ldots, n; \quad \nu = 1, \ldots, \varkappa$$

$$r_\alpha^{\cdot} = 2 r_\alpha \sum_{j=1}^{N} c_{\alpha j} r_j + \ldots \quad \alpha = n+1, \ldots, N$$

$$r_\alpha \varphi_\alpha^{\cdot} = -i\lambda_\alpha r_\alpha + r_\alpha \sum_{j=1}^{N} d_{\alpha j} r_j + \ldots$$

$$R_\nu^2 = \prod_{s=1}^{n} r_s^{|p_{\nu s}|}, \quad \psi_\nu = \sum_{s=1}^{n} p_{\nu s} \varphi_s, \quad i = \sqrt{-1}$$

$$Q_{\nu s}(\psi_\nu) = a_{\nu s} \cos \psi_\nu + b_{\nu s} \sin \psi_\nu$$

$$(Q_{\nu s}(\psi_\nu) \equiv 0, \text{ if } p_{\nu s} = 0)$$

The system obtained from (1.3) by neglecting the terms which were not written out, is called the model system /1/

2. We shall consider the interaction of the fourth-order resonances according to two schemes.

The first interaction scheme

$$\langle P_\nu^*, \Lambda^* \rangle + \langle P_\nu, \Lambda \rangle = 0, \ \nu = 1, \ldots, \varkappa \tag{2.1}$$

is characterized by the presence of a vector component of the eigenvalues $\Lambda^* = (\lambda_1, \ldots, \lambda_{n_0})$ common to all resonance relations. Therefore the integer-valued vector $P_\nu^* = (p_{\nu 1}, \ldots, p_{\nu n_0})$ will have all, essentially non-zero, relatively prime components, and the integer-valued vector $P_\nu = (p_{\nu n_0+1}, \ldots, p_{\nu n})$ will only have a proportion of the non-zero terms so that the products $\langle P_\nu, \Lambda \rangle$ will not contain the common eigenvalues representing the components of the vector $\Lambda = (\lambda_{n_0+1}, \ldots, \lambda_n)$.

The other scheme of interaction is of the "chain" type:

$$\langle P_{\nu-1}^*, \Lambda_{\nu-1}^* \rangle + \langle P_\nu, \Lambda_\nu \rangle = 0, \ \nu = 1, \ldots, \varkappa \tag{2.2}$$

$$\Lambda_\nu = (\lambda_{l_\nu}, \ldots, \lambda_{n_\nu}), \ P_\nu = (p_{\nu l_\nu}, \ldots, p_{\nu n_\nu})$$

$$\Lambda_\nu^* = (\lambda_{n_\nu - k_\nu}, \ldots, \lambda_{n_\nu}), \ P_\nu^* = (p_{\nu n_\nu - k_\nu}, \ldots, p_{\nu n_\nu})$$

$$|P_{\nu-1}^*| + |P_\nu| = 4, \ 0 \leqslant k_\nu \leqslant n_\nu - l_\nu, \ l_0 = 1, \ n_\varkappa = n$$

Every pair of resonance relations contains k_ν common eigenvalues (l_ν, n_ν, k_ν take values from the set $1, \ldots, n$). We note that the eigenvalues $\lambda_{l_\nu}, \ldots, \lambda_{n_\nu - k_\nu}$ appear only in the ν-th resonance relation ($\nu = 1, \ldots, \varkappa$). This also refers to the eigenvalues $\lambda_{n_1-k_1}, \ldots, \lambda_{n_1}$ at $\nu = 1$ and $\lambda_{n_\varkappa - k_\varkappa + 1}, \ldots, \lambda_{n_\varkappa}$ at $\nu = \varkappa$. It can be shown that by virtue of the condition given above, of the lack of other fourth-order resonances and of lower-order resonances, n_0 in (2.1) can only take the values 1 or 2 and we must have $k_\nu = 1$ in (2.2). When investigating further interactions according to scheme (2.1), we shall restrict ourselves to the most important practical case $n_0 = 1$.

To obtain the sufficient conditions of instability and asymptotic stability, we can use in each of the cases considered the Lyapunov function in the form /3/

$$2W = \gamma_1 r_1 + \ldots + \gamma_n r_n + r_{n+1} + \ldots + r_N \tag{2.3}$$

where the first sum represents the integral of the resonant part of the model system (i.e. the system obtained from (1.3)) when $c_{kj} = d_{kj} = 0; k, j = 1, \ldots, N$). This means that the constants γ_s obey the equations

$$\sum_{s=1}^{n} a_{\nu s} \gamma_s = 0, \quad \sum_{s=1}^{n} b_{\nu s} \gamma_s = 0, \quad \nu = 1, \ldots, \varkappa \tag{2.4}$$

Differentiating (2.3) we obtain, by virtue of (1.3), taking (2.4) into account,

$$W^{\cdot} = \sum_{s=1}^{n} \gamma_s r_s \sum_{j=1}^{N} c_{sj} r_j + \sum_{\alpha=n+1}^{N} r_\alpha \sum_{j=1}^{N} c_{\alpha j} r_j + \ldots \tag{2.5}$$

where the terms not written out are of at least the third order of smallness.

In the case of the resonances (1.2) without common frequencies, the model system decomposes into $\varkappa$ independent subsystems, and system (2.4) into $\varkappa$ pairs of independent equations. In the case when every resonance relation contains at least three eigenvalues and the rank of the matrix of the coefficients $a_{\nu s}$, $b_{\nu s}$ for every pair of independent equations composed of (2.4) is equal to two, the problem of stability can be solved using the following lemma proved in /3/ while considering a single resonance of odd order.

Lemma. The necessary and sufficient condition for the system of equations (2.4) to have a strictly positive (negative) solution in γ_s is, that at least one pair of vectors exists for every $\nu = 1, \ldots, \varkappa$

$$a_\nu = (a_{\nu j}, a_{\nu k}, a_{\nu l}),\ b_\nu = (b_{\nu j}, b_{\nu k}, b_{\nu l})$$
$$j, k, l = 1, \ldots, n;\ j \neq k \neq l,\ n \geqslant 3$$

for which there is no change of sign within the series of numbers $D_{kl}^\nu, D_{lj}^\nu, D_{jk}^\nu$, representing the covariant components of the vector product $a_\nu \times b_\nu$.

As was shown in /4/, the conditions of the lemma also remain valid in the case of the resonances conforming to one of the types shown above. Here the highest rank of the matrix of system (2.4) is $2\varkappa$, therefore $n - 2\varkappa$ constants γ_s can be chosen arbitrarily.

Using the lamma and the function (2.3) we can easily obtain, with help of Lyapunov's theorems on instability and asymptotic stability, the sufficient conditions of instability and asymptotic stability.

Theorem 1. Let system (1.3) describe one of the following types of multiple resonance: 1) independent, 2) interacting according to the scheme (2.1) with $n_0 = 1$, 3) interacting according to the scheme (2.2) with $k_\nu = 1$, and let the conditions of the lemma hold for the resonance coefficients of the normal form $a_{\nu s}, b_{\nu s}$. Then, provided that $n - 2\varkappa$ $(n > 2\varkappa)$, the positive constants chosen from $\gamma_1, \ldots, \gamma_n$, and satisfying the equations (2.4), can be set out in such a manner that the quadratic form appearing in (2.5) will be negative definite, and the trivial solution of system (1.3) will be asymptotically stable.

Theorem 2. When fourth-order resonance relations of one of the above types exist, then the trivial solution of system(1.3) will be unstable provided that one of the following conditions holds:

a) the condition of the lemma does not hold for the resonant coefficients, and constants $\gamma_1, \ldots, \gamma_n$, satisfying Eq.(2.4) can be chosen so that the quadratic form appearing in (2.5) is sign definite;

b) the condition of the lemma holds for the resonant coefficients and the quadratic form mentioned above can be made sign definite by choosing $n - 2\varkappa$ negative coefficients for the set $\gamma_1, \ldots, \gamma_n$, satisfying Eqs.(2.4):

c) the condition of the lemma holds for the resonant coefficients, and the quadratic form mentioned above can be made positive definite by choosing $n - 2\varkappa$ positive constants from the set $\gamma_1, \ldots, \gamma_n$, satisfying the equation (2.4).

Note. When $a_{\nu s} = b_{\nu s} = 0$ $(\nu = 1, \ldots, \varkappa;\ s = 1, \ldots, n)$, we have a non-resonant case which was basically considered in /5, 6/, while when $c_{kj} = d_{kj} = 0$ $(k, j = 1, \ldots, N)$, the problem will be solved in the same manner as in the case of the multiple, odd order resonance.

3. We will illustrate the efficiency of the proposed method by considering a special case of system (1.3) $n = N = 3$ for the interacting resonances

$$\lambda_1 + 3\lambda_2 = 0,\quad 3\lambda_1 + \lambda_3 = 0 \tag{3.1}$$

In spite of the fact that the conditions ofthe lemma lose thier meaning here, we can show that in this case the necessary and sufficient conditions for a strictly positive (negative) solution of (2.4) to exist have the form

$$\begin{aligned} &a_{11}b_{12} - a_{12}b_{11} = 0,\quad a_{21}b_{23} - a_{23}b_{21} = 0 \\ &b_{11}b_{12} < 0,\quad b_{21}b_{23} < 0 \end{aligned} \tag{3.2}$$

and the constants γ_1, γ_2 can be expressed in terms of the arbitrary constant γ_3

$$\gamma_1 = -\frac{b_{23}}{b_{21}}\gamma_3,\quad \gamma_2 = \frac{b_{23}}{b_{21}}\frac{b_{11}}{b_{12}}\gamma_3 \tag{3.3}$$

We note that the equations in (3.2) are always realized in Hamiltonian systems, and the inequalities represent the necessary and sufficient conditions for stability of the resonant part of the model system.

The matrix M of the quadratic form from which expansion (2.5) beings can, in the present case, be written in the form

$$M = \| M_{\alpha\beta} \|,\quad M_{\alpha\beta} = \gamma_\alpha c_{\alpha\beta} + \gamma_\beta c_{\beta\alpha};\quad \alpha, \beta = 1, 2, 3 \tag{3.4}$$

Using the Sylvester criterion we obtain the following conditions of negative definiteness of the matrix M:

$$c_{11} < 0, \quad F \equiv -(4b_{11}b_{12}c_{11}c_{12} + \delta_1^2) > 0 \tag{3.5}$$
$$G \equiv \frac{b_{23}}{b_{12}b_{21}^2}\left(\frac{b_{23}}{b_{12}}c_{33}F + \frac{\delta_1\delta_2\delta_3}{b_{12}b_{21}} - \frac{b_{11}}{b_{21}}c_{22}\delta_2^2 + \frac{c_{11}}{b_{12}b_{21}}\delta_3^2\right) < 0$$
$$\delta_1 = b_{11}c_{21} - b_{12}c_{12}, \quad \delta_2 = b_{21}c_{31} - b_{23}c_{13}$$
$$\delta_3 = b_{11}b_{23}c_{23} + b_{12}b_{21}c_{32}$$

Using Theorem 1 we conclude that the inequalities (3.5), conditions (3.2) and $\gamma_3 > 0$ together yield the sufficient conditions for asymptotic stability. The form of the inequalities obtained enables us to conclude that they are consistent. Moreover, in the present case the conditions of asymptotic stability can be obtained without resorting to selecting the value of the arbitrary constant γ_3.

When the signs of the first and last inequality of (3.5) are changed, the trivial solution of (1.3) will, according to Theorem 2 (case c), be unstable. We see that the trivial solution will be unstable whether or not conditions (3.2) hold, as long as the inequalities $F > 0$, $c_{11}G < 0$, are satisfied, since cases a) and b) of Theorem 2 hold.

4. We shall now consider another possible approach to determining the instability of system (1.3) in the case of the interaction of the resonances according to schemes (2.1) and (2.2).

Theorem 3. Let us assume that one of the resonances interacting according to scheme (2.1) or (2.2), with $\nu = \beta$, has its instability exposed by the presence of an invariant ray of the truncated model system of equations. In this system corresponding to (1.3), $c_{sj} = d_{sj} = 0$ for $s = 1, \ldots, n;\ j = n+1, \ldots, N$, and also for those $s, j = 1, \ldots, n$, for which $p_{\beta j} = p_{\beta-1 j} = 0$. Then the trivial solution is unstable, provided that the following condition holds for the remaining resonance relations in the case of interaction (2.1):

$$|P_\nu| > 1, \quad \nu \neq \beta \tag{4.1}$$

while in the case of interaction (2.2) none of the conditions

$$|P^*_{\beta-2}| \leqslant 1, \quad k_{\beta-1} = n_{\beta-1} - l_{\beta-1}, \quad |P_{\beta-1}| \leqslant 1 \tag{4.2}$$

hold.

The validity of this assertion follows from the presence of an increasing particular solution of the model system corresponding to (1.3). The variables corresponding to one of the resonances increases as an invariant ray, and the remaining variables remain zero. This in turn guarantees the instability* (*See Medvedev S.V. A proof of the lemma on instability. Moscow, Dep. v VINITI, 12.3.82, No.1088-82,1982) of the zero solution of the complete system (1.3).

Theorem 3 can be used for an important special case, namely for canonical systems (the normalizing variable transformations shown can be taken in canonical form and are carried out according to the algorithm described in /1/). The case corresponds to the following choice of parameters in (1.3):

$$a_{\nu s} = 0, \quad b_{\nu s} = p_{\nu s}b_\nu \quad (\nu = 1, \ldots, \varkappa;\ s = 1, \ldots, n) \tag{4.3}$$
$$c_{lj} = 0, \quad d_{lj} = d_{jl} \quad (l, j = 1, \ldots, N)$$

Let, for examples, one of the resonances (2.1) be strong when $\nu = \beta$, with the other resonances missing. This, according to /1/, means that the following conditions of the presence of a particular solution in the form of an invariant ray holds:

$$\left|b_\beta p_{\beta 1}^{p_{\beta 1}/2} \ldots p_{\beta n_0}^{p_{\beta n_0}/2} p_{\beta n_0+1}^{p_{\beta n_0+1}/2} \ldots p_{\beta n}^{p_{\beta n}/2}\right| >$$
$$\left|\sum_{k=1}^{n_0}\left(\sum_{l=1}^{n_0} d_{kl}p_{\beta k}p_{\beta l} + \sum_{l=n_0+1}^{n} d_{kl}p_{\beta k}p_{\beta l}\right) + \sum_{k=n_0+1}^{n}\sum_{l=n_0+1}^{n} d_{kl}p_{\beta k}p_{\beta l}\right|$$

Part of the components $p_{\beta l}\ (l = n_0 + 1, \ldots, n)$ of the resonance vector P_ν, corresponding to the frequencies not appearing in the resonance relation chosen, vanishes. Therefore, the corresponding coefficients $d_{kl}\ (k = 1, \ldots, n)$ will not affect the magnitude of the right-hand side of the inequality as they would if these coefficients were missing altogether. This shows clearly the feasibility of applying in this case the first assertion of Theorem 3.

The efficiency of the second assertion of the theorem is proved in the same manner.

Example. Let us consider a photogravitational, circular bounded three-body problem differing from the classical case in the fact that the positively gravitating point is also subjected to the pressure of light from one of the main bodies /7/. The problem admits of seven positions of relative equilibrium, five of which coincide with the known libration points /8/. In /9/, in the region of stability of the sixth and seventh libration point, to a first approximation, an interaction of two strong resonances, $\lambda_1 + 3\lambda_2 = 0$ and $3\lambda_1 + \lambda_3 = 0$.

was found. Taking the second of the resonance relations $(\nu = \beta = 2)$, as the fundamental resonance, we obtain $|P_1| = 3 > 1$. But this means that the case of interaction considered here satisfies all the conditions of Theorem 3; therefore the libration points shown are unstable.

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SMALL VIBRATIONS OF ONE-DIMENSIONAL MOVING BODIES*

V.V. POPOV

Problems of the transverse vibrations of moving strings, hoses with flowing liquid, as well as bodies that can be represented in the form of a set of interacting strings moving at different velocities are examined. It is assumed that there are not tangential stresses between the strings. The vibrations are described by a second-order linear differential equation whose coefficients are obtained by summing the corresponding parameters of the separate strings. The distinctive feature of this kind of system is the difference in the wave propagation velocities in the forward and reverse directions.

A transformation is presented that enables the problem of vibrational processes in a moving body with conditions given on fixed boundaries to be reduced to a boundary value problem for a string at rest. Questions concerning the critical velocities, the free vibration energy of the moving body, and the type of dissipative term are considered. Analytic solutions are given for problems regarding free vibrations and the steady-state regime of forced vibrations under the action of a force varying sinusoidally with time.

1. Formulation of the problem. In a linear approximation we will consider the transverse vibrations of a body (or system of bodies) moving uniformly and rectilinearly along the x axis in the ground state. In the simplest case, the equation of a taut filament (string) moving at a velocity v is obtained from the equations of the string at rest

$$\rho u_{tt} - T u_{xx} = F \tag{1.1}$$

(the notation is standard) by replacing the partial derivative with respect to the time $\partial/\partial t$ by the substantive derivative $\partial/\partial t + v\partial/\partial x$

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